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On $\omega\beta-$Open Sets

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ABSTRACT

In this paper we introduce a new class of sets called $\omega\beta-$open sets which contains the class of $\omega-$open and $\beta-$open sets. Here we carry out a study of the properties of $\omega\beta-$open sets.

Keywords: $\omega-$open, $\beta-$open, $\omega\beta-$open, $\beta-$Lindelöf, $\omega\beta-\mathrm{T}_2$, $\omega\beta-$regular.

1. INTRODUCTION

Throughout this work a space will always mean a topological space with no separation axioms assumed, unless otherwise stated. Let (X,τ) be a space and A a subset of X. By $Int(A)$ and $Cl(A)$, we denote the interior of A and the closure of A, respectively, in (X,τ). A point $x \in X$ is called a condensation point of A if for each open set U containing x, the set $U \cap A$ is uncountable. A is said to be $\omega-$closed (Hdeib (1982)) if it contains all its condensation points. The complement of an $\omega-$closed set is said to be $\omega-$ open. Note that a subset A of a space (X,τ) is $\omega-$ open (Al-Zoubi and Al-Nashef (2003)) if and only if for each $x\in A$ there exists an open set U containing x such that $U-A$ is countable. The family of all $\omega-$ open subsets of a space (X,τ), denoted by $\omega O(X,\tau)$, forms a topology on X, denoted by τ_ω, finer than τ. Several characterizations of $\omega-$ closed subsets were proved in Hdeib (1982).

A subset A of a space (X,τ) is said to be $b-$ open (Andrijević (1996)) (resp. $\beta-$ open (Monesf *et al.* (1983)) or semi-preopen (Andrijević (1986)) if $A\subseteq Int(Cl(A))\bigcap Cl(Int(A))$ (resp. $A\subseteq Cl(Int(Cl(A)))$). The complement of a $\beta-$ open set is called a $\beta-$ closed set. The set of all $\beta-$ open subset in (X,τ) will be denoted by $\beta O(X,\tau)$.

In 2008, Noiri *et al.* introduced the notion of $\omega b-$ open sets. They defined a subset A of a space (X,τ) to be $\omega b-$ open if for every $x\in A$ there exists a $b-$ open set U in (X,τ) containing x such that $U-A$ is countable. The family of all $\omega b-$ open subsets of (X,τ) will be denoted by $\omega bO(X,\tau)$.

In this paper, we introduce the notion of $\omega\beta-$ open sets which is a new generalization of both $\omega-$ open sets and $\beta-$ open sets. We investigate some properties of $\omega\beta-$ open sets. Moreover, by using $\omega\beta-$ open sets, we define and investigate $\beta-$ Lindelöf spaces, $\omega\beta-\mathrm{T}_2$ spaces, $\omega\beta-$ regular spaces, and $\omega\beta-$ normal spaces.

Now we recall some known results which will be used in the sequel.

Lemma 1.1. (Andrijevic (1986)).
If U is open and A is $\beta-$ open in a space (X,τ), then $U\bigcap A$ is $\beta-$ open in (X,τ).

Theorem 1.2. (Navalagi (2002)).
Let (X,τ) be a space, $A\subset Y\subset X$ and Y be $\beta-$ open in (X,τ). Then A is $\beta-$ open in (X,τ) if and only if A is $\beta-$ open in the subspace (Y,τ_Y).

2. $\omega\beta-$OPEN SETS

Definition 2.1.
A subset A of a space (X,τ) is said to be $\omega\beta-$open if for every $x\in A$, there exists a $\beta-$open set U containing x such that $U-A$ is countable. The family of all $\omega\beta-$ open subsets of (X,τ) will be denoted by $\omega\beta O(X,\tau)$.

Note that if X is a countable set then every subset of X is $\omega\beta-$open in (X,τ).

Theorem 2.2.
For any space (X,τ), the following properties hold:

(i) Every $\omega b-$open set is $\omega\beta-$open.
(ii) Every $\beta-$open set is $\omega\beta-$open.

The proof follows easily from Definition 2.1.

$$\begin{array}{ccccc} \text{Open} & \rightarrow & b-\text{open} & \rightarrow & \beta-\text{open} \\ \downarrow & & \downarrow & & \downarrow \\ \omega-\text{open} & \rightarrow & \omega b-\text{open} & \rightarrow & \omega\beta-\text{open} \end{array}$$

The converses need not be true as shown by the following examples.

Example 2.3.
Let $X=\{1,2,3\}$, $\tau=\{X,\phi,\{1\},\{2\},\{1,2\}\}$, then $\{3\}$ is $\omega-$ open in (X,τ), but it is not $\beta-$open.

Example 2.4.
Let $X=\mathbb{R}$ with the usual topology τ_u and let $A=\mathbb{Q}$ be the set of all rational numbers. Then, since $A\subset Int(Cl(A))$, A is $b-$open in (X,τ), but it is not $\omega-$open.

Example 2.5.
Let $X=\mathbb{R}$ with the usual topology τ_u and let $A=[0,1)\cap\mathbb{Q}$. Then, by Lemma 3.5 of Noiri (1984) $Cl(Int(Cl(A)))=Cl[Int(Cl([0,1)))\cap Int(Cl($

$\mathbb{Q}))] = Cl[(0,1) \cap X] = [0,1] \supset A$. Therefore, A is $\beta -$ open, but it is not $\omega b -$ open in (X, τ) by Example 2.10 of Noiri (2008).

Lemma 2.6.
A subset A of a space (X, τ) is $\omega\beta -$ open if and only if for every $x \in A$, there exists a $\beta -$ open set U and a countable subset C such that $x \in U - C \subseteq A$.

Proof.
Let A be $\omega\beta -$ open and $x \in A$. Then there exists a $\beta -$ open set $U \subseteq X$ containing x such that $U - A$ is countable. Hence U is $\beta -$ open, $C = U - A$ is the countable set and $U - C \subseteq A$. The converse is obvious.

Proposition 2.7.
The intersection of an $\omega\beta -$ open set and an $\omega -$ open set is $\omega\beta -$ open.

Proof.
Let A be an $\omega\beta -$ open set and B an $\omega -$ open set. Then there exist a $\beta -$ open set U_A containing x such that $U_A - A$ is countable and an open set U_B containing x such that $U_B - B$ is countable. By Lemma 1.1, $U_A \cap U_B$ is a $\beta -$ open set containing x and $(U_A \cap U_B) - (A \cap B)$ $\subseteq (U_A - A) \cap (U_B - B)$ and hence $(U_A \cap U_B) - (A \cap B)$ is a countable set. Therefore $(A \cap B)$ is an $\omega\beta -$ open set.

Corollary 2.8.
The intersection of an $\omega\beta -$ open set and an open set is $\omega\beta -$ open.

The intersection of a $\beta -$ open set and an $\omega\beta -$ open set is not always $\omega\beta -$ open as the following example shows. Thus, the intersection of two $\omega\beta -$ open sets need not be $\omega\beta -$ open.

Example 2.9.
Let $X = \mathbb{R}$ with the cocountable topology $\tau = \tau_{coc}$. Let $A = (0,1]$ and $B = [1,2)$. Then A and B are $\omega\beta -$ open, but $A \cap B$ is not $\omega\beta -$ open, since each $\beta -$ open containing 1 is uncountable.

Proposition 2.10.
The union of any family of $\omega\beta-$open sets is $\omega\beta-$open.

Proof.
If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of $\omega\beta-$open subsets of X, then for every $x \in \bigcup_{\alpha\in\Delta} A_\alpha$, $x \in A_{\alpha_\circ}$ for some $\alpha_\circ \in \Delta$. Hence there exists a $\beta-$open subset U containing x such that $U - A_{\alpha_\circ}$ is countable. Hence $U - \bigcup_{\alpha\in\Delta} A_\alpha$ is countable. This shows that $\bigcup_{\alpha\in\Delta} A_\alpha$ is $\omega\beta-$open.

Definition 2.11.
The topology generated by $\omega\beta O(X,\tau)$ is defined as $T(\omega\beta O(X,\tau)) = \{V \subseteq X : V \bigcap S \in \omega\beta O(X,\tau)$ whenever $S \in \omega\beta O(X,\tau)\}$.

Proposition 2.12.
$T(\omega\beta O(X,\tau))$ is a topology on X larger than τ.

Proof.
We first show that $T(\omega\beta O(X,\tau))$ is a topology on X. It is clear, by Definition 2.11, $\phi, X \in T(\omega\beta O(X,\tau))$. Now, let $\{U_\alpha : \alpha \in \Delta\}$ be a collection of $\omega\beta-$ open sets in (X,τ), then for each $\alpha \in \Delta$, $U_\alpha \bigcap S$ is $\omega\beta-$open in (X,τ) whenever S is $\omega\beta-$open in (X,τ). By Proposition 2.10, $(\bigcup_{\alpha\in\Delta} U_\alpha) \bigcap S = \bigcup_{\alpha\in\Delta} (U_\alpha \bigcap S) \in \omega\beta O(X,\tau)$ whenever $S \in \omega\beta O(X,\tau)$. Thus by Definition 2.11, $\bigcup_{\alpha\in\Delta} U_\alpha \in T(\omega\beta O(X,\tau))$. Finally, let $U_1, U_2 \in T(\omega\beta O(X,\tau))$ and S be any $\omega\beta-$ open set. Then $U_1 \bigcap S \in \omega\beta O(X,\tau)$ and hence $(U_1 \bigcap U_2) \bigcap S = U_2 \bigcap (U_1 \bigcap S) \in \omega\beta O(X,\tau)$. Therefore, $U_1 \bigcap U_2 \in T(\omega\beta O(X,\tau))$. This proves that $T(\omega\beta O(X,\tau))$ is a topology on X. To show $\tau \subseteq T(\omega\beta O(X,\tau))$, let $U \in \tau$. Then by Corollary 2.8 $U \bigcap S \in \omega\beta O(X,\tau)$ whenever $S \in \omega\beta O(X,\tau)$. Therefore, $U \in T(\omega\beta O(X,\tau))$.

Recall that the topology generated by $\beta O(X,\tau)$ (Ganster and Andrijević (1988)) is defined as $T(\beta O(X,\tau)) = \{V \subseteq X : V \bigcap S \in \beta O (X,\tau)$ whenever $S \in \beta O(X,\tau)$ whenever $S \in \beta O(X,\tau)\}$. In Ganster and Andrijević (1988), $T(\beta O(X,\tau))$ is denoted by T_δ. But, we use the notation $T(\beta O(X,\tau))$ instead of T_δ.

Theorem 2.13.
For a topological space (X,τ), $\omega T(\beta O(X,\tau)) \subseteq T(\omega\beta O(X,\tau))$, the converse is true if every $\beta-$open set is open in (X,τ).

Proof.
Let $A \in \omega T(\beta O(X,\tau))$. We show that $A \bigcap V \in \omega\beta O(X,\tau)$ for every $V \in \omega\beta O(X,\tau)$. Let $x \in A \bigcap V$, then $x \in A$ and $x \in V$. Then there exists $O \in T(\beta O(X,\tau))$ such that $x \in O$ and $O - A$ is countable. And there exists $B \in \beta O(X,\tau)$ such that $x \in B$ and $B - V$ is countable. Since $O \in T(\beta O(X,\tau))$, $x \in O \bigcap B \in \beta O(X,\tau)$. On the other hand, $(O \bigcap B) - (A \bigcap V) \subset (O - A) \bigcup (B - V)$ and $(O - A) \bigcup (B - V)$ is countable. Therefore, $(O \bigcap B) - (A \bigcap V)$ is countable and hence $A \bigcap V \in \omega\beta O(X,\tau)$. Conversely, Let $A \in T(\omega\beta O(X,\tau))$, So $A \bigcap V \in \omega\beta O(X,\tau)$ for every $V \in \omega\beta O(X,\tau)$. Let $x \in A \bigcap V$, then there exists $O \in \beta O(X,\tau)$ containing x such that $O - (A \bigcap V)$ is countable. By assumption, $O \in T(\beta O(X,\tau))$ such that $(O - A)$ is countable. Hence, $A \in \omega T(\beta O(X,\tau))$.

Theorem 2.14.
Let (Y,τ_Y) be a subspace of (X,τ) and $A \subseteq Y$:

(i) If A is $\omega\beta-$open in (X,τ) and Y is open, then A is $\omega\beta-$open in (Y,τ_Y).

(ii) If A is $\omega\beta-$ open in (Y,τ_Y) and Y is $\beta-$ open, then A is $\omega\beta-$open in (X,τ).

Proof.
(i) Let A be $\omega\beta-$open in (X,τ). For every $x \in A$, there exists a $\beta-$open set U in (X,τ) containing x such that $U - A$ is countable. If $U \subseteq Y$ by

Theorem 1.2 U is $\beta-$ open in (Y,τ_Y) containing x and hence A is $\omega\beta-$open in (Y,τ_Y). If U is not contained in Y, set $U_\circ = U\bigcap Y$. Then by Lemma 1.1 $U_\circ$ is $\beta-$open in (Y,τ_Y) and $U_\circ - A$ is countable. Therefore, A is $\omega\beta-$open in (Y,τ_Y).

(ii) Let A be $\omega\beta-$open in (Y,τ_Y). Then for every $x\in A$, there exists a $\beta-$ open set U in (Y,τ_Y) containing x such that $U-A$ is countable. Hence by Theorem 1.2, A is $\omega\beta-$open in (X,τ).

In Theorem 2.14, we can not delate the assumption that Y is $\beta-$open in (X,τ). To see that consider the space $(\mathbb{R},\tau_{coc})$ and the subset $Y=\mathbb{Q}$ of $\mathbb{R}$ and take a subset $A=\{1\}$ of Y. Hence A is $\omega\beta-$open in (Y,τ_Y) but A is not $\omega\beta-$open in $(\mathbb{R},\tau_{coc})$.

Corollary 2.15.
Let (Y,τ_Y) be a subspace of (X,τ), $A\subseteq Y$ and Y be an open set. Then A is $\omega\beta-$open in (X,τ) if and only if A is $\omega\beta-$open in (Y,τ_Y).

Definition 2.16.
A subset F of a topological space (X,τ) is said to be $\omega\beta-$ closed if $X-F$ is $\omega\beta-$open. The family of all $\omega\beta-$closed subsets of (X,τ) will be denoted by $\omega\beta C(X,\tau)$.

Proposition 2.17.
Let (X,τ) be a space and $C\subseteq X$. If C is $\omega\beta-$closed in (X,τ), then $C\subseteq K\bigcap B$ for some $\beta-$closed subset K and a countable subset B.

Proof.
If C is $\omega\beta-$ closed, then $X-C$ is $\omega\beta-$ open and hence for every $x\in X-C$, there exist a $\beta-$ open set U in (X,τ) containing x and a countable set B such that $U-B\subseteq X-C$, thus $C\subseteq (X-U)\bigcup B$. Let $K=X-U$, then K is a $\beta-$closed set such that $C\subseteq K\bigcup B$.

Remark 2.18.
Let (X,τ) be a space and A,B subsets of X. Then the following properties hold.

(i) The intersection of any family of $\omega\beta-$closed sets in (X,τ) is $\omega\beta-$closed.

(ii) Every $\omega\beta-$ closed set and every $\beta-$ closed set in (X,τ) are $\omega\beta-$closed.

(iii) If A is $\omega-$ closed and B is $\omega\beta-$ closed, then $A\bigcup B$ is $\omega\beta-$closed.

The intersection of all $\omega\beta-$closed sets of X containing A is called the $\omega\beta-$closure of A and is denoted by $\omega\beta Cl(A)$. And the union of all $\omega\beta-$open sets of X contained in A is called the $\omega\beta-$interior and is denoted by $\omega\beta Int(A)$.

Theorem 2.19.
Let A be a subset of a space (X,τ). Then $x\in\omega\beta Cl(A)$ if and only if for every $\omega\beta-$open set U containing x, $A\bigcap U\neq\phi$.

Proof.
First, suppose that $x\in\omega\beta Cl(A)$ and U is any $\omega\beta-$open containing x such that $A\bigcap U=\phi$. Then $(X-U)$ is an $\omega\beta-$closed set containing A. Thus, $\omega\beta Cl(A)\subseteq(X-U)$. Then $x\notin\omega\beta Cl(A)$, which is a contradiction. Conversely, Suppose $x\notin\omega\beta Cl(A)$, there exists an $\omega\beta-$closed set V such that $A\subseteq V$ and $x\notin V$. Then $X-V$ is an $\omega\beta-$open set containing x and $A\bigcap(X-V)=\phi$.

3. $\beta-$LINDELÖF SPACES

Definition 3.1.

(i) A space (X,τ) is said to be $\beta-$Lindelöf if every $\beta-$open cover of X has a countable subcover.

(ii) A subset A of a space (X,τ) is said to be $\beta-$Lindelöf relative to X if every cover of A by $\beta-$open sets of (X,τ) has a countable subcover.

Theorem 3.2.
For a space (X,τ), the following properties are equivalent:

(i) (X,τ) is $\beta-$Lindelöf.

(ii) Every $\omega\beta-$open cover of X has a countable subcover.

Proof.
(i) $\rightarrow$ (ii) Let $\{U_\alpha : \alpha \in \Delta\}$ be an $\omega\beta-$open cover of X. For each $x \in X$ there exists $\alpha_{(x)} \in \Delta$ such that $x \in U_{\alpha_{(x)}}$. Since $U_{\alpha_{(x)}}$ is $\omega\beta-$open, there exists a $\beta-$open set $V_{\alpha_{(x)}}$ such that $x \in V_{\alpha_{(x)}}$ and $V_{\alpha_{(x)}} - U_{\alpha_{(x)}}$ is countable. The family $\{V_{\alpha_{(x)}} : x \in X\}$ is a $\beta-$ open cover of X and (X,τ) is $\beta-$ Lindelöf. There exists a countable subset, says $\alpha_{(x_1)}, \alpha_{(x_2)} .... \alpha_{(x_n)}....$ such that $X = \bigcup\{V_{\alpha_{(x_i)}} : i \in \mathbb{N}\}$. Now, we have $X = \{\bigcup_{i\in\mathbb{N}}(V_{\alpha_{(x_i)}} - U_{\alpha_{(x_i)}})\} \bigcup\{\bigcup_{i\in\mathbb{N}} U_{\alpha_{(x_i)}}\}$. For each $\alpha_{(x_i)}$, $V_{\alpha_{(x_i)}} - U_{\alpha_{(x_i)}}$ is a countable set and there exists a countable subset $\Delta_{\alpha_{(x_i)}}$ of Δ such that $V_{\alpha_{(x_i)}} - U_{\alpha_{(x_i)}} \subseteq \bigcup\{U_\alpha : \alpha \in \Delta_{\alpha_{(x_i)}}\}$. Therefore, we have $X \subseteq [\bigcup_{i\in\mathbb{N}}(\bigcup\{U_\alpha : \alpha \in \Delta_{\alpha_{(x_i)}}\}) \bigcup (\bigcup_{i\in\mathbb{N}} U_{\alpha_{(x_i)}})]$.

(ii) $\rightarrow$ (i) The proof is obvious since every $\beta-$open set is $\omega\beta-$open.

A family ζ of subsets of a space (X,τ) is said to have the countable intersection property if the intersection of any countable subcollection of ζ is nonempty.

Theorem 3.3.
For a space (X,τ), the following properties are equivalent

(i) (X,τ) is $\beta-$Lindelöf.

(ii) Every family $\{A_\alpha : \alpha \in \Delta\}$ of $\omega\beta-$ closed sets which has the countable intersection property has a non-empty intersection.

Proof.

(i) $\rightarrow$ (ii) First, let (X,τ) be a $\beta-$Lindelöf space and $\{A_\alpha : \alpha \in \Delta\}$ be a family of $\omega\beta-$closed sets having the countable intersection property. We show that $\bigcap_{\alpha\in\Delta} A_\alpha \neq \phi$. Suppose that $\bigcap_{\alpha\in\Delta} A_\alpha = \phi$, then $\{X - A_\alpha : \alpha \in \Delta\}$ is a collection of $\omega\beta-$open sets and $\bigcup_{\alpha\in\Delta}(X - A_\alpha) = X - \bigcap_{\alpha\in\Delta} A_\alpha = X - \phi = X$. Since (X,τ) is $\beta-$Lindelöf and $\{X - A_\alpha : \alpha \in \Delta\}$ is an $\omega\beta-$open cover of X, there exists a countable subset $\Delta_\circ$ of Δ such that $\bigcup\{X - A_\alpha : \alpha \in \Delta_\circ\} = X$. Therefore $\bigcap_{\alpha\in\Delta_\circ} A_\alpha = \bigcap_{\alpha\in\Delta_\circ}(X-(X - A_\alpha)) = X - (\bigcup_{\alpha\in\Delta_\circ}(X - A_\alpha)) = X - X = \phi$. This contradicts the fact that $\{A_\alpha : \alpha \in \Delta\}$ has the countable intersection property. Therefore $\bigcap_{\alpha\in\Delta} A_\alpha \neq \phi$.

(ii) $\rightarrow$ (i) Let $\{U_\alpha : \alpha \in \Delta\}$ be any $\omega\beta-$ open cover of X. Then $\{X - U_\alpha : \alpha \in \Delta\}$ is a family of $\omega\beta-$closed sets and $\bigcap_{\alpha\in\Delta}(X - U_\alpha) = X - (\bigcup_{\alpha\in\Delta} U_\alpha) = X - X = \phi$. By hypothesis, there is some countable subcollection $X - U_{\alpha_1},...,X - U_{\alpha_n},...$ of this collection such that $\bigcap_{i\in\mathbb{N}}(X - U_{\alpha_i}) = \phi$; hence $\bigcup_{k\in\mathbb{N}} U_{\alpha_k} = \bigcup_{k\in\mathbb{N}}(X-(X - U_{\alpha_k})) = X - \bigcap_{k\in\mathbb{N}}(X - U_{\alpha_k}) = X$. Thus, $U_{\alpha_1},...,U_{\alpha_n},...$ is a countable subcover of $\{U_\alpha : \alpha \in \Delta\}$.

Theorem 3.4.

If (X,τ) is $\beta-$ Lindelöf and B is $\beta-$ closed in (X,τ), then B is $\beta-$Lindelöf relative to X.

Proof.

Let $\{A_\alpha : \alpha \in \Delta\}$ be any cover of B by $\beta-$open sets of X. Since B is $\beta-$closed, $X - B$ is $\beta-$open and $\{A_\alpha : \alpha \in \Delta\}\bigcup(X - B)$ is a $\beta-$open cover of X. Since (X,τ) is $\beta-$Lindelöf, there exists a countable subcover for X, say, $\{A_{\alpha_1}, A_{\alpha_2},...,A_{\alpha_k},...(X - B)\}$. Therefore, $B \subset \bigcup_{k\in\mathbb{N}} A_{\alpha_k}$ and hence $\bigcup_{k\in\mathbb{N}} A_{\alpha_k}$ forms a countable subcover of $\bigcup_{\alpha\in\Delta} A_\alpha$ for B. Therefore, B is $\beta-$Lindelf relative to X.

A topological space (X,τ) is called a $P-$space (Balasubramanian 1982) if every countable intersection of open sets is open in (X,τ).

Theorem 3.5.
Let (X,τ) be a Hausdorff $P-$space and A $\beta-$Lindelöf relative to X. If $x \notin A$, then there exist disjoint open sets U and V containing x and A, respectively.

Proof.
Since X is a T_2-space, for each $a \in A$, there exist disjoint open sets V_a and U_{x_a} containing a and x, respectively. The family $\{V_a : a \in A\}$ forms an open cover of A and hence forms a $\beta-$open cover of A. Since A is $\beta-$ Lindelöf relative to X, there exists a countable subcover $V_{a1}, V_{a2}, ..., V_{an}, ...$. For each $V_{ak}, k = 1,2,3,...,\infty$, there exists a corresponding $U_{x_{ak}}$ and hence $\bigcap_{k=1}^{\infty} U_{x_{ak}} = U$ is open and contains x. But U does not intersect any $V_{ak}, 1 \le k \le \infty$. The reason is that if $U \bigcap V_{ai} \ne \phi$, for some $1 \le i \le \infty$, then $U_{x_{ai}} \bigcap V_{ai} \ne \phi$ since $U \subseteq U_{x_{ak}}, k = 1,2,3,...,\infty$. However, this is contrary to the way $U_{x_{ak}}$ and V_{a_k} were chosen. Thus if we define $V = \bigcup_{k=1}^{\infty} V_{ak}$, then $U \bigcap V = \phi, x \in U$ and $A \subseteq V$.

Theorem 3.6.
Let (X,τ) be a Hausdorff $P-$space. If A is $\beta-$Lindelöf relative to X, then A is closed in (X,τ).

Proof.
Let A be $\beta-$Lindelöf relative to X and $x \notin A$. By Theorem 3.5, there exist open sets U and V containing x and A, respectively, such that $U \bigcap V = \phi$. Thus $U \bigcap A = \phi$. Therefore, $U \subseteq X - A$, which implies $X - A$ is open. Therefore A is closed in (X,τ).

4. SEPARATION AXIOMS

Definition 4.1.
A space (X,τ) is said to be $\omega\beta - T_2$ if for each two distinct points $x, y \in X$, there exist two $\omega\beta-$open sets U and V such that $x \in U, y \in V$ and $U \bigcap V = \phi$.

Note that every T_2-space is $\omega\beta - T_2$ but the converse is not true. For example, consider the space $(\mathbb{N}, \tau_{cof})$, where $\mathbb{N}$ is the set of all natural numbers and τ_{cof} is the cofinite topology. Since every subset of $(\mathbb{N}, \tau_{cof})$ is $\omega\beta-$open, $(\mathbb{N}, \tau_{cof})$ is $\omega\beta - T_2$, but $(\mathbb{N}, \tau_{cof})$ is not a T_2-space.

Theorem 4.2.
For any space (X,τ), the following properties are equivalent:

(i) (X,τ) is $\omega\beta - T_2$.

(ii) Let $x \in X$. For each $y \in X - \{x\}$, there exists an $\omega\beta-$open set U containing x such that $y \notin \omega\beta Cl(U)$.

(iii) For each $x \in X$, $\bigcap\{\omega\beta Cl(U) : U$ is an $\omega\beta-$open set containing $x\} = \{x\}$.

Proof.
(i) $\rightarrow$ (ii) Suppose that (X,τ) is $\omega\beta - T_2$. Then for any distinct points x, y, there exist two disjoint $\omega\beta-$open sets U and V containing x and y, respectively. Thus $U \subseteq X - V$ and hence $\omega\beta Cl(U) \subseteq X - V$. Therefore, $y \notin \omega\beta Cl(U)$.
(ii) $\rightarrow$ (iii) Let $x \in X$ and $y \in X - \{x\}$. Then by (ii) $y \notin \omega\beta Cl(U)$ for some $\omega\beta-$open set U containing x. Therefore, $\bigcap\{\omega\beta Cl(U) : U$ is an $\omega\beta-$open set containing $x\} \subset \{x\}$.
(iii) $\rightarrow$ (i) Suppose that $x, y \in X$ and $x \neq y$. Then there exists an $\omega\beta-$ open set U containing x such that $y \notin \omega\beta Cl(U)$. Now take $V = X - \omega\beta Cl(U)$, then V is $\omega\beta-$ open. Hence $x \in U, y \in V$ and $U \bigcap V = \phi$. Therefore, (X,τ) is $\omega\beta - T_2$.

Definition 4.3.
A space (X,τ) is said to be $\omega\beta-$regular if each pair of a point and a closed set not containing the point can be separated by disjoint $\omega\beta-$open sets.

Theorem 4.4.
For a space (X,τ), the following properties are equivalent:

(i) (X,τ) is $\omega\beta-$regular.

(ii) For each $x\in X$ and each open set U containing x, there exists an $\omega\beta-$open set V such that $x\in V\subseteq \omega\beta Cl(V)\subseteq U$.

(iii) For each closed set F, $\bigcap\{\omega\beta Cl(V):F\subseteq V, V$ is $\omega\beta-$open$\}$ $=F$.

(iv) For each subset A of (X,τ) and each open set U in (X,τ) with $A\bigcap U\neq\phi$, there exists an $\omega\beta-$open set V such that $V\bigcap A\neq\phi$ and $\omega\beta Cl(V)\subseteq U$.

(v) For each non-empty set A and each closed set F with $A\bigcap F=\phi$, there exist two disjoint $\omega\beta-$ open sets U and V such that $A\bigcap U\neq\phi$ and $F\subseteq V$.

Proof.
(i) $\rightarrow$ (ii) Let U be an open set and $x\in U$. Then $X-U$ is closed in X and $x\notin X-U$. By (i), there exist two disjoint $\omega\beta-$open sets V_1 and V_2 such that $(X-U)\subseteq V_1$ and $x\in V_2$. Therefore, $V_2\subseteq (X-V_1)$ and hence $x\in V_2\subseteq \omega\beta Cl(V_2)\subseteq X-V_1\subseteq U$.\\
(ii) $\rightarrow$ (iii) Let F be a closed set and $x\notin F$. By (ii), there exists an $\omega\beta-$open set V such that $x\in V\subseteq \omega\beta Cl(V)\subseteq (X-F)$. Now, we take $U=X-\omega\beta Cl(V)$, then U is $\omega\beta-$ open, $F\subset U$ and $U\bigcup V=\phi$. By Theorem 2.19, $x\notin \omega\beta Cl(U)$. Thus $\bigcap\{\omega\beta C(V):F\subseteq V, V$ is $\omega\beta-$open$\}$ $\subseteq F$. And the converse is obvious.
(iii) $\rightarrow$ (iv) Let A be a subset of X and U be an open set such that $x\in U\bigcap A$. Then $x\notin (X-U)$ and by (iii), there exists an $\omega\beta-$open set V such that $X-U\subseteq V$ and $x\notin \omega\beta Cl(V)$. Now take $M=X-\omega\beta Cl(V)$, then M is an $\omega\beta-$ open set containing x. Thus, $A\bigcap M\neq\phi$ and $M\subseteq X-V$ and hence $\omega\beta Cl(M)\subseteq X-V\subseteq U$.

(iv) $\rightarrow$ (v) Suppose that $A \neq \phi$ and F is a closed set such that $A \bigcap F = \phi$. Then $(X-F)$ is open and $(X-F)\bigcap A \neq \phi$. By (iv), there exists an $\omega\beta-$open set U such that $A \bigcap U \neq \phi$ and $\omega\beta Cl(U) \subseteq (X-F)$. Now if we take $V = X - \omega\beta Cl(U)$, then V is $\omega\beta-$ open, $F \subseteq V$ and $U \bigcap V = \phi$.

(v) $\rightarrow$ (i) Suppose F is a closed set such that $x \in X - F$. By (v), there exist two disjoint $\omega\beta-$open sets U and V such that $x \in U$ and $F \subseteq V$. Thus, (X,τ) is $\omega\beta-$regular.

Definition 4.5.
A space (X,τ) is said to be $\omega\beta-$normal if every two disjoint closed sets can be separated by $\omega\beta-$open sets.

Theorem 4.6.
Aspace (X,τ) is $\omega\beta-$normal if and only if for each closed set F and any open set V containing F, there exists an $\omega\beta-$ open set U such that $F \subseteq U \subset \omega\beta Cl(U) \subseteq V$.

Proof.
NECESSITY. Let F be a closed set and V any open set containing F. Since $(X-V)$ and F are closed sets such that $(X-V)\bigcap F = \phi$, by assumption there exist two disjoint $\omega\beta-$open sets U_1 and U_2 such that $(X-V) \subseteq U_1$ and $F \subseteq U_2$. Since $U_1 \bigcap U_2 = \phi, U_1 \bigcap \omega\beta Cl(U_2) = \phi$ and hence $\omega\beta Cl(U_2) \subseteq X - U_1 \subseteq V$. Thus $F \subseteq U_2 \subseteq \omega\beta Cl(U_2) \subseteq V$.

SUFFICIENCY. Let A_1 and A_2 be any two disjoint closed sets. Now $X - A_2$ is an open set containing A_1 and by assumption there exists an $\omega\beta-$open set B such that $A_1 \subseteq B \subseteq \omega\beta Cl(B) \subseteq X - A_2$. Now we take $U_1 = B$ and $U_2 = X - \omega\beta Cl(B)$, then U_1 and U_2 are disjoint $\omega\beta-$open sets such that $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$. Therefore, (X,τ) is $\omega\beta-$normal.

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